Inadequacy of the usual Newtonian formulation for certain problems in particle mechanics

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The problem of the rigid double pendulum is used to show that the Newtonian axioms are inadequate to obtain a solution. It is further shown that an additional independent axiom, such as the law of moment of momentum, must be introduced to make the solution possible.

I. INTRODUCTION

Although solved correctly some 300 years ago, the problem of the rigid double pendulum is an example which suffices to convince the reader of the need for an additional postulate when dealing with constrained particle motion. It will be shown that the problem of the rigid double pendulum has no solution when Newton's third law is taken to include the centrality (or collinearity) of the internal forces. When centrality is not assumed the "principle of angular momentum" no longer follows as a theorem and must be postulated as an independent law. The application of the law of linear momentum and that of moment of momentum then leads to a noncentral internal force system.

There are a number of other problems which exhibit the same difficulty; e.g., a mass point constrained to slide on the massless rod of a simple pendulum. The analysis here is confined to the rigid double pendulum, for simplicity.

II. NEWTONIAN FORMULATION

By way of background, the main ingredients of the mechanics of a system of particles are now summarized.

Consider a set (or system) $\mathscr{M}$ of m particles P_i, $i = 1,...,m$, with masses m_i, respectively. The position vector of P_i relative to an arbitrary fixed origin in an inertial reference frame is $\mathbf{r}_i = x_i^1\,\mathbf{i}_1 + x_i^2\mathbf{i}_2 + x_i^3\,\mathbf{i}_3$, where (x_i^1, x_i^2, x_i^3) are rectangular Cartesian coordinates and $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ is a fixed orthonormal basis. The velocity and acceleration of P_i at the instant t of time are denoted by $\dot{\mathbf{r}}_i(t)$ and $\ddot{\mathbf{r}}_i(t)$, respectively.

For a system of particles in which all forces are assumed to arise from the interaction of the particles with one another, the forces acting on a subsystem $\mathscr{N} \subseteq \mathscr{M}$, of $n \leqslant m$ particles, may be subdivided into *external* and *internal* forces. The external forces acting on $\mathscr{N}$ are those exerted by particles in $\mathscr{M} \setminus \mathscr{N}$ on particles in $\mathscr{N}$, while the internal forces are those exerted by particles in $\mathscr{N}$ on each other. The external force acting on P_i is denoted by $\mathbf{F}_i$, while the internal force exerted on P_i by P_j is denoted by $\mathbf{F}_{ij}$. Either of these forces may generally be a function of the current positions and velocities of all n particles as well as time.

The usual formulation of Newtonian particle mechanics[1] consists of the following postulates[2]:

(i) Newton's second law in the form

$$\mathbf{F}_i(t) + \sum_{j=1}^{n} \mathbf{F}_{ij}(t) = m_i\ddot{\mathbf{r}}_i(t) \tag{1}$$

for each particle P_i in the system $\mathscr{N}$. Of course, (1) includes Newton's first law for a single particle.

(ii) Newton's third law in the form

$$\mathbf{F}_{ij}(t) = -\mathbf{F}_{ji}(t) \quad \text{and} \quad \mathbf{F}_{ii}(t) = \mathbf{0} \tag{2}$$

for $i = 1,...,n$; $j = 1,...,n$.

(iii) The internal forces $\mathbf{F}_{ij}(t)$ are *central*[3]; that is,

$$[\mathbf{r}_i(t) - \mathbf{r}_j(t)] \times \mathbf{F}_{ij}(t) = \mathbf{0} \tag{3}$$

for $i = 1,...,n$; $j = 1,...,n$.

Furthermore, the masses of the particles are always assumed to be nonzero.

For any given particle P_i in the system the problem to be solved in particle mechanics belongs to one of the following categories:

(a) The postion $\mathbf{r}_i(t)$ of a particle P_i is completely specified as a function of the time t.

(b) The particle P_i is constrained to move on a specified surface or curve.

(c) All of the forces acting on P_i are prescribed as functions of the kinematic variables or some of the forces are prescribed together with classes of permissible velocities or accelerations.

A problem is considered solved when the motion $\mathbf{r}_i(t)$ and the external and internal forces have been obtained as functions of time in such a way as to satisfy Newton's laws (i) and (ii).

On the basis of assumptions (i)–(iii) it is easy to establish the following theorem.

Theorem 1: Moment of momentum. Assume that the forces, masses, and accelerations of a system of particles satisfy the postulates (i)–(iii). Then, $\dot{\mathbf{H}}_0 = \mathbf{M}_0$, where $\mathbf{H}_0$ is the moment of momentum of the subsystem $\mathscr{N} \subseteq \mathscr{M}$ with respect to a point 0 fixed in an inertial reference frame and where $\mathbf{M}_0$ is the total moment about 0 of the external forces acting on $\mathscr{N}$. (See Fig. 1.)

Proof. For convenience, let 0 be the origin. For each particle, the application of Newton's second law (i) implies

$$\mathbf{F}_i(t) + \sum_{j=1}^{n} \mathbf{F}_{ij}(t) = m_i\ddot{\mathbf{r}}_i(t) \tag{4}$$

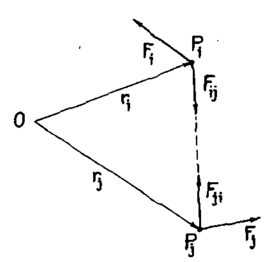

Fig. 1. Central force assumption.

for $i = 1,...,n$. Cross multiplication of each equation by $\mathbf{r}_i(t)$ and a summation over i yield

$$\sum_{i=1}^{n} \mathbf{r}_i(t)\times\mathbf{F}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{r}_i(t)\times\mathbf{F}_{ij}(t)$$

$$= \sum_{i=1}^{n} m_i\mathbf{r}_i(t)\times\ddot{\mathbf{r}}_i(t) . \tag{5}$$

Newton's third law (ii), together with the assumption that the internal forces are central (iii), then implies that

$$\dot{\mathbf{H}}_0(t) = \sum_{i=1}^{n} m_i\mathbf{r}_i(t)\times\ddot{\mathbf{r}}_i(t)$$

$$= \sum_{i=1}^{n} \mathbf{r}_i(t)\times\mathbf{F}_i(t) = \mathbf{M}_0(t) . \tag{6}$$

In applications, Theorem 1 is habitually used as if it were an independent axiom; that is, no check is made as to whether the hypotheses of the theorem are satisfied or not. However, the use of the theorem obviously is justified only if all of the hypotheses are met. As will be shown, the hypotheses are, in fact, not satisfied in the case of the rigid double pendulum. Indeed, the postulates (i)–(iii) are insufficient to solve this problem.

Preliminary to the discussion of the rigid double pendulum, it is instructive to first analyze the simple pendulum in the light of the above remarks.

III. SIMPLE PENDULUM

In the standard treatment of the simple pendulum, the freebody diagram of the particle P is drawn as in Fig. 2, with the tensile force $\mathbf{T} = -T\mathbf{e}_r$ and weight $\mathbf{W} = mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)$ of the particle. An application of Newton's second law yields

$$\ddot{\theta} + (g/a)\sin\theta = 0 \tag{7}$$

as the $\mathbf{e}_\theta$-component equation, and

$$T = mg\cos\theta + ma\dot{\theta}^2 \tag{8}$$

as the $\mathbf{e}_r$-component equation. Equation (7) may be used (subject to appropriate initial conditions) to obtain $\theta(t)$, and (8) then gives the tension required to maintain this motion. It should be emphasized that the tension T was not specified as regards its functional dependence on $\theta(t)$ but rather that this dependence was *derived* through the application of postulates and geometrical constraints. Note that, in addition to Newton's second law, the solution was based on the assumed centrality of the internal force $\mathbf{T}$.

The above problem also serves as an illustration of the use of Theorem 1. With 0 as the fixed point and with $\mathbf{W}$ as the external force, one again obtains (7). Next, Newton's second law, together with the central force assumption, is used to calculate T. Thus, in the case of the simple pendulum, the use of Theorem 1 is justified since a central force

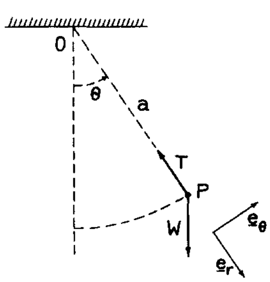

Fig. 2. Standard simple pendulum.

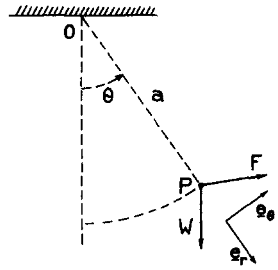

Fig. 3. Variant of the simple pendulum.

does, in fact, suffice to sustain the motion.

Suppose now that one attempted to solve the problem without making use of the central force assumption. To this end, let the internal force on P be represented as $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta$. The use of Newton's second law yields

$$F_\theta = mg\sin\theta + ma\ddot{\theta} , \tag{9}$$

$$F_r = -mg\cos\theta - ma\dot{\theta}^2 . \tag{10}$$

One thus has two equations in the three unknowns θ, F_θ, and F_r, so that an additional condition must be introduced before a solution is possible. (See Fig. 3.)

As one such condition, the central force assumption, $F_\theta = 0$, would again yield (7) and (8). Alternatively, one may assume $\dot{\mathbf{H}}_0 = \mathbf{M}_0$ as an independent axiom. With $\mathbf{F}$ considered as an internal force, and with $\mathbf{W}$ as the external force, one obtains

$$ma^2\ddot{\theta} = -mga\sin\theta . \tag{11}$$

By substituting (11) into (9), one *deduces* that $F_\theta = 0$, and F_r may be determined as before.

It follows that in this problem, the central force assumption and an independent law of moment of momentum are equivalent in the sense that one may be deduced from the other once Newton's second law is assumed. This equivalence does not hold in general, however, as the next example demonstrates.

IV. RIGID DOUBLE PENDULUM

In this section, a simple problem in the constrained motion of particles is presented which *cannot* be solved with the central force assumption but for which an independently postulated axiom of moment of momentum does lead to a solution. It thus appears that the latter postulate allows solutions for a class of problems for which the former postulate fails to provide one.

Problem. Consider a system of three particles P_0,P_1,P_2 having masses m_0,m_1,m_2, respectively, and connected by means of a single massless rod. The particle P_0 is fixed[4] and the rod is hinged at 0 in such a way that the system may move as a pendulum in the xy plane, as illustrated in Fig. 4. The only external forces acting on the system are the weights of the particles and the reaction force at 0. The rod

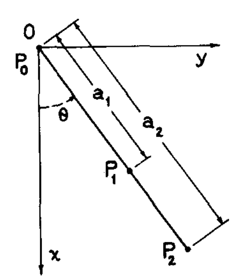

Fig. 4. Rigid double pendulum.

is initially at rest in the horizontal ($0y$) position. Obtain the angular displacement θ as a function of the time.

Strictly speaking, the simple pendulum and the problem as posed above do not conform to the Newtonian formulation in Sec. II, because of the inclusion of such concepts as strings and rods. However, these idealizations are artifices which serve to make the pendulum models physically plausible; they are not essential to the problem statement. In order to cast the pendulum problems within the framework of particle mechanics, the rods and strings must be replaced by suitable statements about the internal forces $\mathbf{F}_{ij}$ and by geometrical constraints on the motion.

Consider then the rigid double pendulum as posed within the framework of Sec. II. It is idealized as a system of three particles P_0, P_1,P_2 aligned along a straight line for all time and moving on concentric circles. The particles are acted upon by the external forces comprised of the weights of the particles, $\mathbf{W}_0$,$\mathbf{W}_1$,$\mathbf{W}_2$, acting vertically downward, and the reaction $\mathbf{F}_0$ at 0. In addition, the action of the connecting members upon the particles is idealized as a set of internal forces $\{\mathbf{F}_{10},\mathbf{F}_{01},\mathbf{F}_{20},\mathbf{F}_{02},\mathbf{F}_{12},\mathbf{F}_{21}\}$. The unknown reaction, the internal forces and the motion are to be determined.

Ideally, any axioms postulated for the solution of dynamical problems provide necessary and sufficient conditions for that solution; that is, for the determination of the motion and the forces (or possibly, their resultants). For a given set of unknowns it is generally accepted that there must be at least as many conditions available for their determination as there are unknowns. With reference to the rigid double pendulum, the process of the introduction of unknowns and the provision of an equal number of conditions for their determination will now be illustrated.

The solution begins with the postulation of Newton's second law (i) for each particle as the first relationship between the unknowns. Based on the freebodies of Fig. 5, one has the equations of motion given by

$$\begin{aligned} &\mathbf{W}_0 + \mathbf{F}_0 + \mathbf{F}_{01} + \mathbf{F}_{02} = m_0\ddot{\mathbf{r}}_0 \,, \\ &\mathbf{W}_1 + \mathbf{F}_{10} + \mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 \,, \\ &\mathbf{W}_2 + \mathbf{F}_{21} + \mathbf{F}_{20} = m_2\ddot{\mathbf{r}}_2 \,. \end{aligned} \tag{12}$$

The set of vector unknowns in this system of equations is

$$\{\mathbf{F}_0,\mathbf{F}_{01},\mathbf{F}_{10},\mathbf{F}_{02},\mathbf{F}_{20},\mathbf{F}_{12}\mathbf{F}_{21},\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}_2\},$$

so that there are 20 unknown component functions of time for whose determination one has 6 component equations in (12).

The geometric constraints are introduced next. For the particle P_0 one has $\mathbf{r}_0 = \mathbf{0}$; the presumed circular motion of the particle P_1, namely, $\mathbf{r}_1 = a_1\mathbf{e}_r$, yields $\ddot{\mathbf{r}}_1 = a_1(-\dot{\theta}^2\mathbf{e}_r + \ddot{\theta}\mathbf{e}_\theta)$ as the expression for the acceleration; finally, since the particle P_2 remains aligned with P_1, one has $\mathbf{r}_2 = (a_2/a_1)\mathbf{r}_1$. The use of these 5 conditions leaves the unknowns

$$\{\mathbf{F}_0,\mathbf{F}_{01},\mathbf{F}_{10}\mathbf{F}_{,02},\mathbf{F}_{20},\mathbf{F}_{12},\mathbf{F}_{21},\theta\}\,,$$

when suitable initial conditions have been prescribed. Thus there now are 15 unknowns and 6 equations.

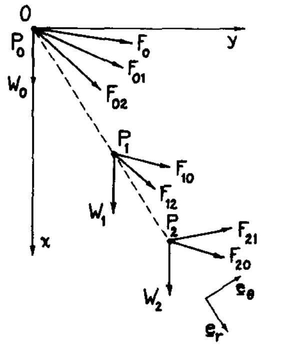

Fig. 5. Rigid double pendulum as a constrained particle motion.

An application of Newton's third law (ii) leaves equations of motion of the form

$$\begin{aligned} &\mathbf{W}_0 + \mathbf{F}_0 - \mathbf{F}_{10} - \mathbf{F}_{20} = \mathbf{0} \,, \\ &\mathbf{W}_1 + \mathbf{F}_{10} + \mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 \,, \\ &\mathbf{W}_2 - \mathbf{F}_{12} + \mathbf{F}_{20} = m_2(a_2/a_1)\ddot{\mathbf{r}}_1 \,, \end{aligned} \tag{13}$$

with the set of unknowns reduced to $\{\mathbf{F}_0,\mathbf{F}_{10},\mathbf{F}_{20},\mathbf{F}_{12},\theta\}$, leaving 9 unknowns and 6 equations.

Evidently, in the Newtonian analysis, the effect of the internal forces on a given particle consists of a resultant internal force due to all of the other particles. Conversely, this resultant is all that one may expect to determine when nothing concerning the functional form of the internal forces has been specified. With this in mind, Eq. (13) may be written as

$$\begin{aligned} &\mathbf{W}_0 + \mathbf{F}_0 - \mathbf{R}_1 - \mathbf{R}_2 = \mathbf{0} \,, \\ &\mathbf{W}_1 + \mathbf{R}_1 = m_1\ddot{\mathbf{r}}_1, \quad \mathbf{R}_1 = \mathbf{F}_{10} + \mathbf{F}_{12} \,, \\ &\mathbf{W}_2 + \mathbf{R}_2 = m_2(a_2/a_1)\ddot{\mathbf{r}}_1, \quad \mathbf{R}_2 = -\mathbf{F}_{12} + \mathbf{F}_{20} \,. \end{aligned} \tag{14}$$

The new set of unknowns is $\{\mathbf{F}_0,\mathbf{R}_1,\mathbf{R}_2,\theta\}$. With the weights written as

$$\mathbf{W}_i = m_i\, g(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \quad i = 0,1,2, \tag{15}$$

the system (14) of 6 equations in 7 unknowns may now be expressed in the form

$$\begin{aligned} \mathbf{F}_0 = &-[(m_1a_1 + m_2a_2)\dot{\theta}^2 + (m_0 + m_1 + m_2)\, g\cos\theta\,]\mathbf{e}_r \\ &+ [(m_1a_1 + m_2a_2)\ddot{\theta} + (m_0 + m_1 + m_2)\, g\sin\theta\,]\mathbf{e}_\theta \,, \\ \mathbf{R}_i = &-m_i(a_i\dot{\theta}^2 + g\cos\theta\,)\mathbf{e}_r \\ &+ m_i(a_i\ddot{\theta} + g\sin\theta\,)\mathbf{e}_\theta \quad i = 1,2 \,. \end{aligned} \tag{16}$$

An additional condition is needed to solve this system of equations. Usually, this condition is taken to be the assumed centrality of the internal forces, condition (iii) above. When used here, this condition comprises two separate conditions requiring that the $\mathbf{e}_\theta$ components of $\mathbf{R}_1$ and $\mathbf{R}_2$ vanish. It then follows from (16) that

$$\ddot{\theta} = -(g/a_1)\sin\theta \quad \text{and} \quad \ddot{\theta} = -(g/a_2)\sin\theta \,. \tag{17}$$

Since a_2 is not equal to a_1 in general, the $\mathbf{e}_\theta$ components of $\mathbf{R}_1$ and $\mathbf{R}_2$ cannot vanish simultaneously and one is led to the conclusion that the internal force system cannot be central. It follows that *this problem has no solution within the traditional framework of Newtonian dynamics.*

Suppose now that instead of assuming centrality of the internal forces, one postulates an independent law of moment of momentum. Thus applying (6), one obtains

$$\ddot{\theta} = -\frac{m_1a_1 + m_2a_2}{m_1a_1^2 + m_2a_2^2}\, g\sin\theta \tag{18}$$

and hence, by integration,

$$\dot{\theta}^2 = \frac{2(m_1a_1 + m_2a_2)}{m_1a_1^2 + m_2a_2^2}\, g\cos\theta \,, \tag{19}$$

where the inital conditions $\dot{\theta}(0) = 0$ and $\theta(0) = \pi/2$ have

been used.

The corresponding forces are then given by

$$\mathbf{F}_0 = -\frac{g\cos\theta}{m_1a_1^2+m_2a_2^2}\,[(m_0+m_1+m_2)(m_1a_1^2+m_2a_2^2) + 2(m_1a_1+m_2a_2)^2]\mathbf{e}_r + \frac{g\sin\theta}{m_1a_1^2+m_2a_2^2} \times[(m_0+m_1+m_2)(m_1a_1^2+m_2a_2^2) - (m_1a_1+m_2a_2)^2]\mathbf{e}_\theta \quad (20)$$

and

$$\mathbf{R}_i = -\frac{m_i g\cos\theta}{m_1a_1^2+m_2a_2^2}\,[m_1a_1^2+m_2a_2^2 + 2a_i(m_1a_1+m_2a_2)]\mathbf{e}_r + \frac{m_i g\sin\theta}{m_1a_1^2+m_2a_2^2} \times[m_1a_1^2+m_2a_2^2 - a_i(m_1a_1+m_2a_2)]\mathbf{e}_\theta \quad i=1,2\,. \quad (21)$$

This internal force system clearly is not central.

V. CONCLUSIONS

The preceding example shows that there are simple problems in constrained particle motion for which the traditional Newtonian axioms are insufficient to obtain a solution. It has also been shown that by invoking the law of moment of momentum (as a separate postulate) the problem of the rigid double pendulum became soluble.

It is of interest to note that for the present problem, the postulation of an independent law of moment of momentum is equivalent to the following postulates in the sense that Eq. (18) is the consequence of their application:

(a) The law of conservation of kinetic and potential energy. A choice of zero potential energy at a vertical distance a_2 from 0 yields

$$V(t) = m_1 g(a_2 - a_1\cos\theta) + m_2 g a_2(1-\cos\theta) + m_0 g a_2\,,$$
$$T(t) = \tfrac{1}{2}(m_1a_1^2+m_2a_2^2)\dot\theta^2,$$

for the potential and the kinetic energy, respectively. The conservation of the total energy is expressed by $V(t) + T(t) = \text{const}$. A differentiation with respect to t results in Eq. (18).

(b) The system of internal forces is in static equilibrium. This, of course, may be expressed in terms of either the virtual work of the forces or in terms of a summation of forces and moments.

The total virtual work of the system of internal forces vanishes. With admissible virtual displacements given by $\delta\mathbf{r}_0 = \mathbf{0}$, $\delta\mathbf{r}_1 = a_1\delta\theta\mathbf{e}_\theta$, and $\delta\mathbf{r}_2 = a_2\delta\theta\mathbf{e}_\theta$, this statement takes on the form

$$\mathbf{R}_1\cdot\delta\mathbf{r}_1 + \mathbf{R}_2\cdot\delta\mathbf{r}_2 = [(m_1a_1^2+m_2a_2^2)\ddot\theta + (m_1a_1+m_2a_2)g\sin\theta]\delta\theta = 0\,.$$

Since this must be true for any $\delta\theta$, the expression in brackets must vanish.

The summation of internal forces and the summation of their moments with respect to any fixed point are zero. With $\mathbf{R}_0 = \mathbf{F}_{01} + \mathbf{F}_{02}$ as the internal force acting on the particle P_0 one has

$$\mathbf{R}_0 + \mathbf{R}_1 + \mathbf{R}_2 = \mathbf{0}$$

and

$$\tilde{\mathbf{M}}_0 = \mathbf{r}_1\times\mathbf{R}_1 + \mathbf{r}_2\times\mathbf{R}_2 = \mathbf{0}\,.$$

The latter equation again results in Eq. (18).

In closing, it may be remarked that within the framework of the dynamics of deformable media one does, in fact, postulate Euler's second law of motion, the law of moment of momentum, as an independent axiom. It is also interesting to note that from a historical viewpoint, the inception of the law of moment of momentum preceded the Newtonian postulates. The historical development is traced by Truesdell.[2]

ACKNOWLEDGMENTS

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[1] I. Newton, *Philosophiae Naturalis Principia Mathematica.* The printing thereof was completed by S. Pepys on 5 July 1686; the book was published in London in 1687. The most commonly cited English translation is that given in 1729 by Andrew Motte as revised by Florian Cajori: Sir Isaac Newton's *Mathematical Principles of Natural Philosophy* and his *The System of the World* (University of California, Berkeley, CA, 1960).

[2] Reference 1, Vol. 1, p. 13. Newton's laws have been stated here in the usual modern form. They cannot be found precisely in this form anywhere in the *Principia*, although all the basic ideas are there. If Newton's second law is postulated about a *system* of particles, that is, the total external force is set equal to the total rate of change of the linear momentum of the system, and if it is assumed that the law also applies to *every* subsystem, then Newton's third law in the form (ii) follows as a theorem. If Newton's second law is taken as a statement about the motion of an individual particle, as is done here, then the third law does not follow as a theorem but must be independently postulated. For further critiques and historical comments concerning Newton's laws, the works of I. B. Cohen [Texas Quart. X (3), 127–157 (1967)], C. A. Truesdell [*Essays in the History of Mechanics* (Springer-Verlag, New York, 1968)], and I. Szabó [*Geschichte der Mechanischen Prinzipien* (Birkhäuser, Stuttgart, 1977)] are recommended.

[3] Sometimes (ii) and (iii) together are put forward as a *strong* form of Newton's third law; but it seems to be more in keeping with Newton's own statement and usage to regard (ii) alone as representing the third law. [See Ref. 1, *The System of the World*, p. 570.]

[4] The particle P_0 has been included only to conform to the Newtonian idea that forces act on masses.